Midterm – Numerical Simulation of Turbulence

All problems are weighted equally in the calculation of the grade.

1. The vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$ is the curl of the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$: $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. The rate-of-strain tensor $\boldsymbol{S}$ is constructed by symmetrizing the velocity gradient: $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.
   - (a) Show that $\boldsymbol{S}\boldsymbol{\omega} = \boldsymbol{0}$ for any two-dimensional flow.
   - (b) Demonstrate that, in incompressible flow, a non-singular rate-of-strain tensor $\boldsymbol{S}$ necessarily possesses at least one positive and one negative eigenvalue.
   - (c) Show that the Fourier coefficient $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa})$ of the vorticity $\boldsymbol{\omega}$ is $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa}) = \mathrm{i}\boldsymbol{\kappa} \times \hat{\boldsymbol{u}}(\boldsymbol{\kappa})$, where $\hat{\boldsymbol{u}}(\boldsymbol{\kappa})$ is the Fourier coefficient of the velocity $\boldsymbol{u}$.
   - (d) Show that, in an incompressible flow, $\boldsymbol{\kappa}$, $\hat{\boldsymbol{u}}(\boldsymbol{\kappa})$ and $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa})$ are mutually orthogonal.

2. The evolution equation for the kinetic energy $\hat{E}(\kappa, t)$ in Fourier mode κ is given by

$$\frac{d}{dt}\hat{E}(\kappa, t) = \hat{T}(\kappa, t) - 2\nu\kappa^2\hat{E}(\kappa, t),$$

   where ν is the viscosity of the fluid.
   - (a) Explain the two components in the above right-hand side in terms of their physical meaning.
   - (b) Illustrate the idea of the energy cascade in turbulence with reference to the above equation.
   - (c) Kolmogorov's 1941 (K41) turbulence theory is based on two core assumptions. Formulate these assumptions.

3. In large-eddy simulation (LES), a spatial filter is introduced. In this case, we focus on the box filter applied in a single spatial dimension. The filtered velocity is then given by

$$\overline{u}(x, t) = \frac{1}{\Delta}\int_{x-\Delta/2}^{x+\Delta/2} u(\xi, t)d\xi,$$

   where u denotes the velocity and Δ is the filter length.
   - (a) Show that

$$\frac{\partial \overline{u}}{\partial x} = \overline{\frac{\partial u}{\partial x}}.$$

(b) Show that the transfer function of this filter is given by

$$\frac{\sin(\kappa\Delta/2)}{\kappa\Delta/2}$$

(c) Assume $\kappa\Delta$ is small and compute the first two non-zero terms of the Taylor expansion of the above transfer function about $\kappa\Delta = 0$.

(d) Using result (c), demonstrate that for small $\kappa\Delta$:

$$\overline{u} \approx u - \frac{\Delta^2}{24}\frac{\partial^2 u}{\partial x^2}.$$

4. The velocity field $u(x,t)$ satisfies the incompressible Navier–Stokes equations. A spatial filter is applied to this field, yielding the filtered velocity $\overline{u}$. The corresponding fluctuation, or residual component, is defined as $u' = u - \overline{u}$.

   (a) State the (incompressible) Navier-Stokes equations.

   (b) Derive the LES equations for the filtered velocity $\overline{u}$ from (a).

   (c) Outline the closure problem in LES.